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Estimates for solutions to a class of time-delay systems of neutral type with periodic coefficients and several delays

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Abstract. We consider a class of nonlinear time-delay systems of neutral type with periodic coefficients in linear terms and several delays. We establish conditions under which the zero solution is exponentially stable and obtain estimates characterizing exponential decay of solutions at infinity. The conditions are formulated in terms of differential matrix inequalities. All the values characterizing the decay rate are written out in explicit form.

Keywords: systems of neutral type, periodic coefficients, several delays, exponential stability, Lyapunov–Krasovskii functional.

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1 Introduction

There is a large number of works devoted to delay differential equations (for instance, see [1,3,15,17–23,28] and the bibliography therein). One of the important questions is asymptotic stability of solutions to delay differential equations. This question is very important from theoretical and practical viewpoints because delay differential equations arise in many applied problems when describing the processes whose speeds are defined by present and previous states (for example, see [16,24,25] and the bibliography therein).

This article presents a continuation of our works on asymptotic stability of solutions to delay differential equations with periodic coefficients (for instance, see [5,6,11,26]). We consider the system of nonlinear delay differential equations

$$\begin{aligned}\frac{d}{dt}\left(y(t)+Dy(t-\tau_1)\right) &= A(t)y(t)+\sum_{j=1}^{m}B_j(t)y(t-\tau_j)\\ &\quad + F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m)), \qquad t>0,\end{aligned} \tag{1.1}$$

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where D is a constant $(n \times n)$-matrix, $A(t)$, $B_j(t)$ are $(n \times n)$-matrices with continuous T-periodic entries; i.e.,

$$A(t+T) \equiv A(t), \quad B_j(t+T) \equiv B_j(t), \quad j = 1, \dots, m,$$

$\tau_j > 0$ are time delays, $j = 1, \dots, m$, $\tau_1 > \tau_k > 0$, $k = 2, \dots, m$, and $F(t, u, v_1, \dots, v_m)$ is a real-valued vector function satisfying the Lipschitz condition with respect to u and the inequality

$$\|F(t, u, v_1, \dots, v_m)\| \leq q_0 \|u\| + \sum_{j=1}^{m} q_j \|v_j\|, \quad q_j \geq 0, \ j = 0, \dots, m. \tag{1.2}$$

In the case of $D \neq 0$ this system is called one of neutral type [15]. Here and hereafter we use the following dot product and vector norm

$$\langle x, z \rangle = \sum_{j=1}^{n} x_j \bar{z}_j, \qquad \|x\| = \sqrt{\langle x, x \rangle},$$

the symbol $\|D\|$ means the spectral norm of the matrix D.

Our aims are to establish conditions under which the zero solution is exponentially stable and obtain estimates characterizing exponential decay of solutions at infinity. To establish conditions of stability, researchers often use various Lyapunov or Lyapunov–Krasovskii functionals. At present, there is a large number of works in this direction; for example, see the bibliographies in the survey [2] and in the book [30] devoted wholly to obtaining conditions of stability by the use of Lyapunov–Krasovskii functionals. However, not every Lyapunov–Krasovskii functional makes it possible to obtain estimates characterizing exponential decay of solutions at infinity. In recent years, the study in this direction has developed rapidly. In the case of constant coefficients, there are a lot of works for linear delay differential equations including equations of neutral type (for example, see [20] and the bibliography therein).

The case of nonlinear equations with variable coefficients in linear terms is of special interest and is more complicated in comparison with the case of linear equations. Along with estimates of exponential decay of solutions, a very important question is deriving estimates of attraction sets for nonlinear equations. The natural problem is to obtain such estimates by means of Lyapunov–Krasovskii functionals used for exponential stability analysis of equations defined by their linear part. To the best of our knowledge, in the case of constant coefficients, the first constructive estimates were established in [5,6,27]. For periodic coefficients, the first constructive estimates of attraction sets for the system

$$\frac{d}{dt} y(t) = A(t)y(t) + B(t)y(t-\tau) + F(t, y(t), y(t-\tau)) \tag{1.3}$$

using a Lyapunov–Krasovskii functional associated with the exponentially stable linear system

$$\frac{d}{dt} y(t) = A(t)y(t) + B(t)y(t-\tau) \tag{1.4}$$

were obtained in [6].

To study asymptotic stability of solutions to (1.4) with T-periodic coefficients in [5] the authors proposed to use the Lyapunov–Krasovskii functional

$$\langle H(t)y(t), y(t) \rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s) \rangle \, ds, \tag{1.5}$$

where the matrix valued functions $H \in C(\overline{\mathbb{R}}_+) \cap C^1([lT,(l+1)T])$, $l = 0,1,\dots$, $K \in C^1([0,\tau])$ are such that

$$H(t) = H^*(t), \quad H(t) = H(t+T) > 0, \qquad t \geq 0, \tag{1.6}$$

$$K(s) = K^*(s), \quad K(s) > 0, \quad \frac{d}{ds}K(s) < 0, \qquad s \in [0,\tau]. \tag{1.7}$$

Here and hereinafter the matrix inequality $Q > 0$ (or $Q < 0$) means that the Hermitian matrix Q is positive (or negative) definite. In the case of the T-periodic matrix $A(t)$ such that the zero solution to the system of ordinary differential equations

$$\frac{d}{dt}x = A(t)x, \qquad t > 0,$$

is asymptotically stable, it is not difficult to construct the functional (1.5) by the use of the asymptotic stability criterion of the authors' article [4]. Indeed, in accord with this criterion the following boundary value problem for the Lyapunov differential equation

$$\begin{cases} \dfrac{d}{dt}H + HA(t) + A^*(t)H = -Q(t), & t \in [0,T], \\ H(0) = H(T) > 0, \end{cases} \tag{1.8}$$

is uniquely solvable for every continuous matrix $Q(t)$; moreover, if $Q(t) = Q^*(t) > 0$ then $H(t) = H^*(t) > 0$ on $[0,T]$. Extend T-periodically the matrix $H(t)$ to the whole half-axis $\{t > 0\}$ and use it in (1.5), since (1.6) are fulfilled. In view of [5, 6] solutions to (1.4) are asymptotically stable if there exists a matrix $K(s)$ satisfying (1.7) and such that the matrix

$$\begin{pmatrix} Q(t) - K(0) & -H(t)B(t) \\ -B^*(t)H(t) & K(\tau) \end{pmatrix}, \qquad t \in [0,T],$$

is positive definite. Note that this condition is equivalent to the matrix inequality

$$K(0) + H(t)B(t)(K(\tau))^{-1}B^*(t)H(t) < Q(t), \qquad t \in [0,T].$$

Obviously, for a wide class of T-periodic matrices $B(t)$, the matrix $K(s)$ can be found in the form

$$K(s) = \alpha(s)K_0, \quad K_0 = K_0^* > 0, \quad \text{where} \quad \alpha(s) > 0, \quad \alpha'(s) < 0, \quad s \in [0,\tau].$$

The usage of the functional (1.5) allowed us to obtain estimates of exponential decay of solutions to the linear system (1.4). The authors considered in [6, 26] nonlinear systems of delay differential equations of the form (1.3), where

$$\|F(t,u,v)\| \leq q_1\|u\|^{1+\omega_1} + q_2\|v\|^{1+\omega_2}, \quad q_1 \geq 0,\ q_2 \geq 0,\ \omega_1 \geq 0,\ \omega_2 \geq 0.$$

Using the same functional (1.5), conditions of asymptotic stability of the zero solution were obtained, estimates characterizing the decay rate at infinity were established, and estimates of attraction sets of the zero solution were derived. It should be noted that the estimates are constructive. All the values characterizing the decay rate and attraction sets depend on the matrices $H(t)$ and $K(s)$. As was mentioned above, to construct these matrices it is sufficient to solve the boundary value problem (1.8) for the Lyapunov differential equation with periodic coefficients. The authors in [4] showed that this problem is well-conditioned from the viewpoint of perturbation theory. Therefore we may apply numerical methods for solving this problem to a high degree of accuracy. A survey of computational methods for continuous-time periodic systems can be found in [29]. Thus, the proposed approach makes it possible to study numerically exponential stability of solutions to time-delay systems with periodic coefficients in linear terms.

To study exponential stability of solutions to the systems of linear differential equations of neutral type with constant coefficients, the first author in [7] introduced the Lyapunov–Krasovskii functional

$$\langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s),y(s)\rangle\, ds, \tag{1.9}$$

where $H = H^* > 0$ and the matrix $K(s)$ satisfies (1.7). Using this functional, the study of exponential stability of solutions to systems of the form (1.1) with constant coefficients and one delay was conducted in [7,8,10,14]. There, conditions of exponential stability of the zero solution, estimates of exponential decay of solutions at infinity, and estimates of attraction sets of the zero solution were obtained. Some examples given in [14] show effectiveness of the proposed approach.

The usage of the functionals (1.5) and (1.9) leads to the idea of constructing the Lyapunov–Krasovskii functional

$$\langle H(t)(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s),y(s)\rangle \tag{1.10}$$

for the study of exponential stability of solutions to the linear time-delay system of neutral type with periodic coefficients

$$\frac{d}{dt}(y(t)+Dy(t-\tau)) = A(t)y(t)+B(t)y(t-\tau), \quad t>0. \tag{1.11}$$

Using this functional, the authors in [11] established conditions of exponential stability of the zero solution to (1.11) and derived estimates characterizing exponential decay of solutions at infinity.

In this article we consider the nonlinear time-delay system (1.1) with several delays. As was mentioned above, our aims are to establish conditions of exponential stability of the zero solution to (1.1) and to obtain estimates characterizing exponential decay of solutions to (1.1) at infinity. It should be noted that, in the case of constant coefficients, similar results were established in [9,12,13]. In Sections 2, 3 we study the linear time-delay system

$$\frac{d}{dt}(y(t)+Dy(t-\tau_1)) = A(t)y(t)+\sum_{j=1}^{m} B_j(t)y(t-\tau_j), \qquad t>0. \tag{1.12}$$

We formulate the main results for (1.12) in Section 2 and prove them in Section 3. Using these results, we formulate the main results for (1.1) in Section 4 and prove them in Section 5.

2 Main results for (1.12)

In this section we consider the linear time-delay system (1.12). As was mentioned above, the case of the system with one delay ($m=1$) was studied in [11]. Hereafter we consider $m \geq 2$.

Theorem 2.1. *Suppose that there exist $(n \times n)$-matrix valued functions $H \in C^1([0,T])$, $K_j \in C^1([0,\tau_j])$, $j = 1,\dots,m$:*

$$H(t) = H^*(t), \quad t \in [0,T], \qquad H(0) = H(T) > 0, \tag{2.1}$$

$$K_j(s) = K_j^*(s), \qquad s \in [0,\tau_j], \tag{2.2}$$

$$K_j(s) > 0, \quad \frac{d}{ds}K_j(s) < 0, \qquad s \in [0,\tau_j], \tag{2.3}$$

such that the matrix

$$C(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) \\ C_{12}^*(t) & C_{22}(t) & C_{23}(t) \\ C_{13}^*(t) & C_{23}^*(t) & C_{33}(t) \end{pmatrix} \tag{2.4}$$

with

$$\begin{aligned}
C_{11}(t) &= -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - \sum_{j=1}^{m} K_j(0), \\
C_{12}(t) &= -\frac{d}{dt}H(t)D - H(t)B_1(t) - A^*(t)H(t)D, \\
C_{13}(t) &= \begin{pmatrix} -H(t)B_2(t) & \cdots & -H(t)B_m(t) \end{pmatrix}, \\
C_{22}(t) &= -D^*\frac{d}{dt}H(t)D - D^*H(t)B_1(t) - B_1^*(t)H(t)D + K_1(\tau_1), \\
C_{23}(t) &= \begin{pmatrix} -D^*H(t)B_2(t) & \cdots & -D^*H(t)B_m(t) \end{pmatrix}, \\
C_{33}(t) &= \begin{pmatrix} K_2(\tau_2) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & K_m(\tau_m) \end{pmatrix}
\end{aligned}$$

is positive definite for $t \in [0,T]$. Then the zero solution to (1.12) is exponentially stable.

Remark 2.2. In the case of $m = 1$, the matrix $C(t)$ defined by (2.4) should be replaced with the matrix (see [11])

$$\begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{12}^*(t) & C_{22}(t) \end{pmatrix}.$$

Consider the initial value problem for (1.12)

$$\begin{aligned}
\frac{d}{dt}(y(t) + Dy(t-\tau_1)) &= A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(t-\tau_j), \quad t > 0, \\
y(t) &= \varphi(t), \quad t \in [-\tau_1, 0], \\
y(+0) &= \varphi(0),
\end{aligned} \tag{2.5}$$

where $\varphi(t) \in C^1([-\tau_1,0])$ is a given vector function. Assuming that the conditions of Theorem 2.1 are satisfied, below we provide estimates characterizing exponential decay of solutions to (2.5) as $t \to \infty$. To formulate the results we introduce some notations. If the matrix $H(t)$ satisfies the conditions of Theorem 2.1, then

$$\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) < -\sum_{j=1}^{m} K_j(0);$$

i.e., $H(t)$ is a solution to the boundary value problem (1.8) with $Q(t) = Q^*(t) > 0$. In this case $H(t) > 0$ on $[0,T]$ (see [4]). Extend T-periodically this matrix to the whole half-axis $\{t \geq 0\}$, keeping the same notation. Using this matrix $H(t)$ and the matrices $K_j(s)$, $j = 1,\dots,m$, satisfying the conditions of Theorem 2.1, we define

$$V_0(\varphi) = \langle H(0)(\varphi(0) + D\varphi(-\tau_1)), (\varphi(0) + D\varphi(-\tau_1))\rangle + \sum_{j=1}^{m} \int_{-\tau_j}^{0} \langle K_j(-s)\varphi(s), \varphi(s)\rangle ds \tag{2.6}$$

and

$$\begin{aligned} P(t) = &- \frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - \sum_{j=1}^{m} K_j(0) \\ &- \left(H(t)A(t)D + \sum_{j=1}^{m} K_j(0)D - H(t)B_1(t) \right) \left[K_1(\tau_1) - D^* \sum_{j=1}^{m} K_j(0)D \right]^{-1} \\ &\times \left(D^*A^*(t)H(t) + \sum_{j=1}^{m} D^*K_j(0) - B_1^*(t)H(t) \right) - \sum_{j=2}^{m} H(t)B_j(t)K_j^{-1}(\tau_j)B_j^*(t)H(t). \end{aligned} \tag{2.7}$$

It is not hard to verify that the matrix $P(t)$ is positive definite if the matrix $C(t)$ in (2.4) is positive definite (for details, see Section 3). Denote by $p_{\min}(t) > 0$ the minimal eigenvalue of the matrix $P(t)$ and by $h_{\min}(t)$ the minimal eigenvalue of the matrix $H(t)$. As was mentioned above, $H(t) > 0$. Consequently, $h_{\min}(t) > 0$. Let $k_j > 0$ be the maximal number such that

$$\frac{d}{ds}K_j(s) + k_j K_j(s) \leq 0, \quad s \in [0, \tau_j], \quad j = 1,\dots,m. \tag{2.8}$$

We put

$$\gamma(t) = \min\{p_{\min}(t),\ k_1\|H(t)\|, \dots,\ k_m\|H(t)\|\}, \tag{2.9}$$

$$\Phi = \max_{t\in[-\tau_1,0]} \|\varphi(t)\|, \quad \alpha = \max_{t\in[0,T]} \sqrt{\frac{V_0(\varphi)}{h_{\min}(t)}}, \tag{2.10}$$

$$\beta(t) = \frac{\gamma(t)}{2\|H(t)\|}, \quad \beta^+ = \max_{t\in[0,T]} \beta(t), \quad \beta^- = \min_{t\in[0,T]} \beta(t). \tag{2.11}$$

It is not hard to show that the spectrum of the matrix D belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ if the conditions of Theorem 2.1 are fulfilled; i.e., if the matrix $C(t)$ is positive definite. Hence, $\|D^j\| \to 0$ as $j \to \infty$. Let l be the minimal positive integer such that $\|D^l\| < 1$. In dependence on $\|D^l\|$, below in Theorems 2.3–2.5 we establish estimates if

$$\|D^l\| < e^{-l\beta^+\tau_1}, \qquad e^{-l\beta^+\tau_1} \leq \|D^l\| \leq e^{-l\beta^-\tau_1}, \qquad e^{-l\beta^-\tau_1} < \|D^l\| < 1,$$

respectively.

Theorem 2.3. *Let the conditions of Theorem 2.1 be satisfied and*

$$\|D^l\| < e^{-l\beta^+\tau_1}.$$

Then a solution to the initial value problem (2.5) satisfies the estimate

$$\begin{aligned} \|y(t)\| \leq \Bigg[&\alpha\left(1 - \|D^l\|e^{l\beta^+\tau_1}\right)^{-1} \sum_{j=0}^{l-1} \|D^j\|e^{j\beta^+\tau_1} \\ &+ \max\left\{\|D\|e^{\beta^+\tau_1}, \dots, \|D^l\|e^{l\beta^+\tau_1}\right\} \Phi \Bigg] e^{-\int_0^t \beta(\xi)\,d\xi}, \qquad t > 0, \end{aligned}$$

where α, $\beta(t)$, β^+, and Φ are defined in (2.10) and (2.11).

Theorem 2.4. *Let the conditions of Theorem 2.1 be satisfied and*

$$e^{-l\beta^+\tau_1} \le \|D^l\| \le e^{-l\beta^-\tau_1}.$$

Then a solution to the initial value problem (2.5) satisfies the estimate

$$\|y(t)\| \le \Bigg[\alpha\left(1+\frac{t}{l\tau_1}\right)\sum_{j=0}^{l-1}\|D^j\|e^{j\beta^+\tau_1} \\ + \max\left\{1, \|D\|e^{\beta^+\tau_1}, \dots, \|D^{l-1}\|e^{(l-1)\beta^+\tau_1}\right\}\Phi\Bigg] e^{-\int_0^t \sigma(\xi)\,d\xi}, \qquad t>0,$$

where α, $\beta(t)$, β^+, β^-, and Φ are defined in (2.10) and (2.11),

$$\sigma(t) = \min\left\{\beta(t),\ -\frac{1}{l\tau_1}\ln\|D^l\|\right\}.$$

Theorem 2.5. *Let the conditions of Theorem 2.1 be satisfied and*

$$e^{-l\beta^-\tau_1} < \|D^l\| < 1.$$

Then a solution to the initial value problem (2.5) satisfies the estimate

$$\|y(t)\| \le \Bigg[\alpha\|D^l\|e^{l\beta^-\tau_1}\left(\|D^l\|e^{l\beta^-\tau_1}-1\right)^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta^-\tau_1} \\ + \|D^l\|^{\frac{1}{l}-1}\max\left\{1, \|D\|, \dots, \|D^{l-1}\|\right\}\Phi\Bigg]\exp\left(\frac{t}{l\tau_1}\ln\|D^l\|\right), \qquad t>0,$$

where α, β^-, and Φ are defined in (2.10) and (2.11).

We prove Theorems 2.3–2.5 in Section 3. Obviously, Theorem 2.1 immediately follows from these theorems.

3 Proof of the main results for (1.12)

First, we formulate the auxiliary lemma of the matrix theory used by us further. Here and hereafter we denote by I the unit matrix.

Lemma 3.1. *Let*

$$Q(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{12}^*(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{13}^*(t) & Q_{23}^*(t) & Q_{33}(t) \end{pmatrix}, \qquad t\in[0,T],$$

be a Hermitian positive definite matrix with continuous entries. Then the representation holds

$$Q(t) = \begin{pmatrix} I & \widetilde{Q}_1(t)\widetilde{Q}_2^{-1}(t) & Q_{13}(t)Q_{33}^{-1}(t) \\ 0 & I & Q_{23}(t)Q_{33}^{-1}(t) \\ 0 & 0 & I \end{pmatrix}$$
$$\times \begin{pmatrix} Q_{11}(t) - \widetilde{Q}_1(t)\widetilde{Q}_2^{-1}(t)\widetilde{Q}_1^*(t) - Q_{13}(t)Q_{33}^{-1}(t)Q_{13}^*(t) & 0 & 0 \\ 0 & \widetilde{Q}_2(t) & 0 \\ 0 & 0 & Q_{33}(t) \end{pmatrix}$$
$$\times \begin{pmatrix} I & 0 & 0 \\ \widetilde{Q}_2^{-1}(t)\widetilde{Q}_1^*(t) & I & 0 \\ Q_{33}^{-1}(t)Q_{13}^*(t) & Q_{33}^{-1}(t)Q_{23}^*(t) & I \end{pmatrix},$$

where

$$\widetilde{Q}_1(t) = Q_{12}(t) - Q_{13}(t)Q_{33}^{-1}(t)Q_{23}^*(t), \qquad \widetilde{Q}_2(t) = Q_{22}(t) - Q_{23}(t)Q_{33}^{-1}(t)Q_{23}^*(t);$$

moreover, the matrices $Q_{11}(t) - \widetilde{Q}_1(t)\widetilde{Q}_2^{-1}(t)\widetilde{Q}_1^*(t) - Q_{13}(t)Q_{33}^{-1}(t)Q_{13}^*(t)$, $\widetilde{Q}_2(t)$, *and* $Q_{33}(t)$ *are positive definite.*

To prove Theorems 2.3–2.5 we need the auxiliary results obtained below.

Lemma 3.2. *Let the conditions of Theorem 2.1 be satisfied. Then, for a solution to the initial value problem (2.5), the following inequality holds*

$$\|y(t) + Dy(t-\tau_1)\| \leq \sqrt{\frac{V_0(\varphi)}{h_{\min}(t)}} \exp\left(-\int_0^t \frac{\gamma(\xi)}{2\|H(\xi)\|} d\xi \right), \quad t > 0, \tag{3.1}$$

where $V_0(\varphi)$ *and* $\gamma(t)$ *are defined by (2.6) and (2.9), respectively,* $h_{\min}(t) > 0$ *is the minimal eigenvalue of the matrix* $H(t)$.

Proof. We follow the strategy in [5]. Let $y(t)$ be a solution to the initial value problem (2.5). Using the above matrices $H(t)$ and $K_j(s)$, $j = 1, \dots, m$, we consider the Lyapunov–Krasovskii functional

$$V(t,y) = \langle H(t)(y(t) + Dy(t-\tau_1)), (y(t) + Dy(t-\tau_1)) \rangle + \sum_{j=1}^m \int_{t-\tau_j}^t \langle K_j(t-s)y(s), y(s) \rangle \, ds. \tag{3.2}$$

Clearly, this Lyapunov–Krasovskii functional is a generalization of (1.10) for several delays. The time derivative of this functional is

$$\begin{aligned} \frac{d}{dt}V(t,y) \equiv &\left\langle \frac{d}{dt}H(t)(y(t) + Dy(t-\tau_1)),\ (y(t) + Dy(t-\tau_1)) \right\rangle \\ &+ \left\langle H(t)(A(t)y(t) + \sum_{j=1}^m B_j(t)y(t-\tau_j)),\ (y(t) + Dy(t-\tau_1)) \right\rangle \\ &+ \left\langle H(t)(y(t) + Dy(t-\tau_1)),\ (A(t)y(t) + \sum_{j=1}^m B_j(t)y(t-\tau_j)) \right\rangle \\ &+ \sum_{j=1}^m \langle K_j(0)y(t), y(t) \rangle - \sum_{j=1}^m \langle K_j(\tau_j)y(t-\tau_j), y(t-\tau_j) \rangle \\ &+ \sum_{j=1}^m \int_{t-\tau_j}^t \left\langle \frac{d}{dt}K_j(t-s)y(s), y(s) \right\rangle ds. \end{aligned}$$

Using the matrix $C(t)$ defined by (2.4), we obtain

$$\begin{aligned} \frac{d}{dt}V(t,y) \equiv & -\left\langle C(t)\begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle \\ & + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt}K_j(t-s)y(s), y(s) \right\rangle ds, \end{aligned} \tag{3.3}$$

where

$$z(t) = \begin{pmatrix} y(t-\tau_2) \\ \vdots \\ y(t-\tau_m) \end{pmatrix}.$$

Consider the first summand in the right-hand side of (3.3). Since

$$\begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} = \begin{pmatrix} I & -D & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}$$

then

$$\begin{aligned} & \left\langle C(t)\begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle \\ & \equiv \left\langle S(t)\begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle, \end{aligned} \tag{3.4}$$

where

$$S(t) = \begin{pmatrix} I & 0 & 0 \\ -D^* & I & 0 \\ 0 & 0 & I \end{pmatrix} C(t) \begin{pmatrix} I & -D & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} S_{11}(t) & S_{12}(t) & S_{13}(t) \\ S_{12}^*(t) & S_{22}(t) & S_{23}(t) \\ S_{13}^*(t) & S_{23}^*(t) & S_{33}(t) \end{pmatrix}, \tag{3.5}$$

$$\begin{aligned} & S_{11}(t) = C_{11}(t), \quad S_{12}(t) = C_{12}(t) - C_{11}(t)D, \quad S_{13}(t) = C_{13}(t), \\ & S_{22}(t) = D^*C_{11}(t)D - C_{12}^*(t)D - D^*C_{12}(t) + C_{22}(t), \\ & S_{23}(t) = C_{23}(t) - D^*C_{13}(t), \quad S_{33}(t) = C_{33}(t). \end{aligned}$$

Taking into account the entries of the matrix $C(t)$ in (2.4), we have

$$\begin{aligned} & S_{11}(t) = -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - \sum_{j=1}^{m} K_j(0), \\ & S_{12}(t) = H(t)A(t)D + \sum_{j=1}^{m} K_j(0)D - H(t)B_1(t), \\ & S_{13}(t) = \begin{pmatrix} -H(t)B_2(t) & \cdots & -H(t)B_m(t) \end{pmatrix}, \quad S_{22}(t) = K_1(\tau_1) - D^* \sum_{j=1}^{m} K_j(0)D, \\ & S_{23}(t) = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}, \quad S_{33}(t) = \begin{pmatrix} K_2(\tau_2) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & K_m(\tau_m) \end{pmatrix}. \end{aligned} \tag{3.6}$$

Obviously, the matrix $C(t)$ is positive definite if and only if the matrix $S(t)$ is positive definite. Since $S_{23}(t)$ is the zero matrix, it follows from Lemma 3.1 that

$$
\begin{aligned}
S(t) = &\begin{pmatrix} I & S_{12}(t)S_{22}^{-1}(t) & S_{13}(t)S_{33}^{-1}(t) \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\
&\times \begin{pmatrix} S_{11}(t) - S_{12}(t)S_{22}^{-1}(t)S_{12}^{*}(t) - S_{13}(t)S_{33}^{-1}(t)S_{13}^{*}(t) & 0 & 0 \\ 0 & S_{22}(t) & 0 \\ 0 & 0 & S_{33}(t) \end{pmatrix} \\
&\times \begin{pmatrix} I & 0 & 0 \\ S_{22}^{-1}(t)S_{12}^{*}(t) & I & 0 \\ S_{33}^{-1}(t)S_{13}^{*}(t) & 0 & I \end{pmatrix};
\end{aligned}
$$

moreover, the matrices

$$
P(t) = S_{11}(t) - S_{12}(t)S_{22}^{-1}(t)S_{12}^{*}(t) - S_{13}(t)S_{33}^{-1}(t)S_{13}^{*}(t), \quad S_{22}(t), \quad \text{and} \quad S_{33}(t)
$$

are positive definite. Hence,

$$
\begin{aligned}
&\left\langle S(t) \begin{pmatrix} y(t) + Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t) + Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle \\
&\qquad \geq \langle P(t)(y(t) + Dy(t-\tau_1)), (y(t) + Dy(t-\tau_1)) \rangle .
\end{aligned} \tag{3.7}
$$

Taking into account (3.6), the matrix $P(t)$ has the form (2.7). Consequently, in view of (3.7), from (3.4) we derive

$$
\begin{aligned}
&- \left\langle C(t) \begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle \\
&\qquad \leq - \langle P(t)(y(t) + Dy(t-\tau_1)), (y(t) + Dy(t-\tau_1)) \rangle \\
&\qquad \leq - p_{\min}(t) \|y(t) + Dy(t-\tau_1)\|^2,
\end{aligned} \tag{3.8}
$$

where $p_{\min}(t) > 0$ is the minimal eigenvalue of $P(t)$. Using the matrix $H(t)$, we have

$$
\|y(t) + Dy(t-\tau_1)\|^2 \geq \frac{1}{\|H(t)\|} \langle H(t)(y(t) + Dy(t-\tau_1)), (y(t) + Dy(t-\tau_1)) \rangle .
$$

By (3.8), from (3.3) we obtain

$$
\begin{aligned}
\frac{d}{dt} V(t,y) \leq & - \frac{p_{\min}(t)}{\|H(t)\|} \langle H(t)(y(t) + Dy(t-\tau_1)), (y(t) + Dy(t-\tau_1)) \rangle \\
& + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt} K_j(t-s) y(s), y(s) \right\rangle ds.
\end{aligned}
$$

Using (2.8), we have

$$
\begin{aligned}
\frac{d}{dt} V(t,y) \leq & - \frac{p_{\min}(t)}{\|H(t)\|} \langle H(t)(y(t) + Dy(t-\tau_1)), (y(t) + Dy(t-\tau_1)) \rangle \\
& - \sum_{j=1}^{m} k_j \int_{t-\tau_j}^{t} \langle K_j(t-s) y(s), y(s) \rangle ds.
\end{aligned}
$$

Taking into account the definition of the functional (3.2), we obtain

$$\frac{d}{dt}V(t,y) \le -\frac{\gamma(t)}{\|H(t)\|}V(t,y),$$

where $\gamma(t) = \min\{p_{\min}(t), k_1\|H(t)\|, \dots, k_m\|H(t)\|\}$. From this differential inequality we derive the estimate

$$V(t,y) \le V_0(\varphi)\exp\left(-\int\limits_0^t \frac{\gamma(\xi)}{\|H(\xi)\|}d\xi\right),$$

where $V_0(\varphi)$ is defined by (2.6). Clearly,

$$\|y(t)+Dy(t-\tau_1)\|^2 \le \frac{1}{h_{\min}(t)}\langle H(t)(y(t)+Dy(t-\tau_1)),\ (y(t)+Dy(t-\tau_1))\rangle,$$

where $h_{\min}(t)$ is the minimal eigenvalue of $H(t)$. Then, using the definition of the functional (3.2), we have

$$\|y(t)+Dy(t-\tau_1)\| \le \sqrt{\frac{V(t,y)}{h_{\min}(t)}} \le \sqrt{\frac{V_0(\varphi)}{h_{\min}(t)}}\exp\left(-\int\limits_0^t \frac{\gamma(\xi)}{2\|H(\xi)\|}d\xi\right).$$

The lemma is proved. □

Lemma 3.3. *Let the conditions of Theorem 2.1 be satisfied. Then a solution to the initial value problem (2.5) on every segment $t \in [k\tau_1, (k+1)\tau_1)$, $k = 0, 1, \dots$, satisfies the following estimate*

$$\|y(t)\| \le \alpha \sum_{j=0}^{k} \|D^j\| e^{-\int_0^{t-j\tau_1}\beta(\xi)\,d\xi} + \|D^{k+1}\|\Phi, \tag{3.9}$$

where α, $\beta(t)$, and Φ are defined in (2.10) and (2.11).

Proof. By Lemma 3.2, a solution to the initial value problem (2.5) satisfies (3.1). Taking into account the notations (2.10) and (2.11), we obtain

$$\|y(t)+Dy(t-\tau_1)\| \le \alpha e^{-\int_0^t \beta(\xi)\,d\xi}, \qquad t > 0. \tag{3.10}$$

Obviously, for $t \in [0, \tau_1)$ we have the inequality

$$\|y(t)\| \le \alpha e^{-\int_0^t \beta(\xi)\,d\xi} + \|Dy(t-\tau_1)\| \le \alpha e^{-\int_0^t \beta(\xi)\,d\xi} + \|D\|\Phi,$$

which gives us (3.9) for $k = 0$.

Let $t \in [k\tau_1, (k+1)\tau_1)$, $k = 1, 2\dots$. It is not hard to write out the sequence of the inequalities

$$\begin{aligned}
\|y(t)\| &\le \alpha e^{-\int_0^t \beta(\xi)\,d\xi} + \|Dy(t-\tau_1)\| \\
&\le \alpha e^{-\int_0^t \beta(\xi)\,d\xi} + \|Dy(t-\tau_1) + D^2y(t-2\tau_1)\| + \|D^2y(t-2\tau_1) + D^3y(t-3\tau_1)\| + \dots \\
&\quad + \|D^k y(t-k\tau_1) + D^{k+1}y(t-(k+1)\tau_1)\| + \|D^{k+1}y(t-(k+1)\tau_1)\| \\
&\le \alpha e^{-\int_0^t \beta(\xi)\,d\xi} + \|D\|\,\|y(t-\tau_1) + Dy(t-2\tau_1)\| + \|D^2\|\,\|y(t-2\tau_1) + Dy(t-3\tau_1)\| + \dots \\
&\quad + \|D^k\|\,\|y(t-k\tau_1) + Dy(t-(k+1)\tau_1)\| + \|D^{k+1}\|\,\|y(t-(k+1)\tau_1)\|.
\end{aligned}$$

By (3.10) we derive the estimate

$$\|y(t)\| \leq \alpha e^{-\int_0^t \beta(\xi)\,d\xi} + \alpha\|D\|e^{-\int_0^{t-\tau_1} \beta(\xi)\,d\xi} + \alpha\|D^2\|e^{-\int_0^{t-2\tau_1} \beta(\xi)\,d\xi} + \dots$$
$$+ \alpha\|D^k\|e^{-\int_0^{t-k\tau_1} \beta(\xi)\,d\xi} + \|D^{k+1}\|\Phi,$$

which implies (3.9).

The lemma is proved. □

Proofs of Theorems 2.3–2.5. In the case of one delay, in [11] the analogs of Theorems 2.3–2.5 (see Theorems 2–4 in [11]) were proved in detail by the use of the auxiliary assertions (see Lemmas 2–4 in [11]). In the present paper, using Lemmas 3.2, 3.3 and repeating the reasoning carried out when proving Theorems 2–4 in [11], we derive the required estimates for solutions to the initial value problem (2.5). □

Using the proof of Lemma 3.2, we can reformulate the conditions of exponential stability of the zero solution to the system (1.12) as follows.

Theorem 3.4. *Suppose that there exist $H(t)$, $K_j(s)$, $j = 1,\dots,m$, satisfying (2.1)–(2.3) and such that $\left(K_1(\tau_1) - D^* \sum_{j=1}^m K_j(0)D\right) > 0$ and $P(t)$ defined by (2.7) is positive definite for $t \in [0,T]$. Then the zero solution to (1.12) is exponentially stable.*

4 Main results for (1.1)

In this section we consider the nonlinear time-delay system (1.1). Using the results of Sections 2, 3, we establish conditions of exponential stability of the zero solution to (1.1) and obtain estimates characterizing exponential decay of solutions to (1.1) at infinity.

Theorem 4.1. *Let the conditions of Theorem 2.1 be satisfied, $m \geq 2$,*

$$q(t) = \left(q_0 + \sqrt{q_0^2 + (q_0\|D\| + q_1)^2 + \sum_{j=2}^m q_j^2}\right) \|H(t)\|, \tag{4.1}$$

let $S(t)$ be defined by (3.5), (3.6). Suppose that q_j, $j = 0,\dots,m$, in (1.2) are such that the matrix $(S(t) - q(t)I)$ is positive definite for $t \in [0,T]$. Then the zero solution to (1.1) is exponentially stable.

Remark 4.2. In the case of $m = 1$, the function $q(t)$ defined by (4.1) and the matrix $S(t)$ in (3.5) should be replaced with

$$\left(q_0 + \sqrt{q_0^2 + (q_0\|D\| + q_1)^2}\right)\|H(t)\| \quad \text{and} \quad \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{12}^*(t) & S_{22}(t) \end{pmatrix},$$

respectively.

Consider the initial value problem for (1.1)

$$\begin{aligned} \frac{d}{dt}(y(t) + Dy(t-\tau_1)) &= A(t)y(t) + \sum_{j=1}^m B_j(t)y(t-\tau_j) \\ &\quad + F(t, y(t), y(t-\tau_1), \dots, y(t-\tau_m)), \quad t > 0, \\ y(t) &= \varphi(t), \quad t \in [-\tau_1, 0], \\ y(+0) &= \varphi(0), \end{aligned} \tag{4.2}$$

where $\varphi(t) \in C^1([-\tau_1, 0])$ is a given vector function. This problem has a unique solution because the vector function $F(t, u, v_1, \dots, v_m)$ satisfies the Lipschitz condition with respect to u and $\tau_j > 0$, $j = 1, \dots, m$ (for example, see [15, Ch. 1]). Assuming that the conditions of Theorem 4.1 are satisfied, below we establish estimates characterizing the rate of exponential decay of the solution as $t \to \infty$.

We introduce the matrix

$$\begin{aligned}
\widetilde{P}(t) = & -\frac{d}{dt}H(t) - H(t)A(t) - A^*(t)H(t) - \sum_{j=1}^{m} K_j(0) - q(t)I \\
& - \left(H(t)A(t)D + \sum_{j=1}^{m} K_j(0)D - H(t)B_1(t) \right) \left[K_1(\tau_1) - D^* \sum_{j=1}^{m} K_j(0)D - q(t)I \right]^{-1} \\
& \times \left(D^*A^*(t)H(t) + \sum_{j=1}^{m} D^*K_j(0) - B_1^*(t)H(t) \right) \\
& - \sum_{j=2}^{m} H(t)B_j(t) \left[K_j(\tau_j) - q(t)I \right]^{-1} B_j^*(t)H(t).
\end{aligned} \tag{4.3}$$

It is not hard to verify that the matrix $\widetilde{P}(t)$ is positive definite if the matrix $(S(t) - q(t)I)$ is positive definite (for details, see Section 5). Denote by $\widetilde{p}_{\min}(t) > 0$ the minimal eigenvalue of the matrix $\widetilde{P}(t)$. We put

$$\widetilde{\gamma}(t) = \min \left\{ \widetilde{p}_{\min}(t),\ k_1 \|H(t)\|, \dots,\ k_m \|H(t)\| \right\}, \tag{4.4}$$

$$\widetilde{\beta}(t) = \frac{\widetilde{\gamma}(t)}{2\|H(t)\|}, \quad \widetilde{\beta}^+ = \max_{t \in [0,T]} \widetilde{\beta}(t), \quad \widetilde{\beta}^- = \min_{t \in [0,T]} \widetilde{\beta}(t). \tag{4.5}$$

As was mentioned in Section 2, the spectrum of the matrix D belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ if the conditions of Theorem 2.1 are fulfilled. Hence, $\|D^j\| \to 0$ as $j \to \infty$. Let l be the minimal positive integer such that $\|D^l\| < 1$. In dependence on $\|D^l\|$, we distinguish three cases and establish estimates for solutions to (4.2) if

$$\|D^l\| < e^{-l\widetilde{\beta}^+\tau_1}, \qquad e^{-l\widetilde{\beta}^+\tau_1} \le \|D^l\| \le e^{-l\widetilde{\beta}^-\tau_1}, \qquad e^{-l\widetilde{\beta}^-\tau_1} < \|D^l\| < 1,$$

respectively.

Theorem 4.3. *Let the conditions of Theorem 4.1 be satisfied and*

$$\|D^l\| < e^{-l\widetilde{\beta}^+\tau_1}.$$

Then a solution to the initial value problem (4.2) satisfies the estimate

$$\begin{aligned}
\|y(t)\| \le & \left[\alpha \left(1 - \|D^l\| e^{l\widetilde{\beta}^+\tau_1}\right)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\widetilde{\beta}^+\tau_1} \right. \\
& \left. + \max\left\{ \|D\| e^{\widetilde{\beta}^+\tau_1}, \dots, \|D^l\| e^{l\widetilde{\beta}^+\tau_1} \right\} \Phi \right] e^{-\int_0^t \widetilde{\beta}(\xi)\, d\xi}, \qquad t > 0,
\end{aligned}$$

where α, $\widetilde{\beta}(t)$, $\widetilde{\beta}^+$, and Φ are defined in (2.10) and (4.5).

Theorem 4.4. *Let the conditions of Theorem 4.1 be satisfied and*

$$e^{-l\widetilde{\beta}^{+}\tau_1} \le \|D^l\| \le e^{-l\widetilde{\beta}^{-}\tau_1}.$$

Then a solution to the initial value problem (4.2) satisfies the estimate

$$\|y(t)\| \le \Bigg[\alpha\left(1+\frac{t}{l\tau_1}\right)\sum_{j=0}^{l-1}\|D^j\|e^{j\widetilde{\beta}^{+}\tau_1} + \max\left\{1, \|D\|e^{\widetilde{\beta}^{+}\tau_1}, \dots, \|D^{l-1}\|e^{(l-1)\widetilde{\beta}^{+}\tau_1}\right\}\Phi\Bigg] e^{-\int_0^t \widetilde{\sigma}(\xi)\,d\xi}, \qquad t>0,$$

where α, $\widetilde{\beta}(t)$, $\widetilde{\beta}^{+}$, $\widetilde{\beta}^{-}$, and Φ are defined in (2.10) and (4.5),

$$\widetilde{\sigma}(t) = \min\left\{\widetilde{\beta}(t),\ -\frac{1}{l\tau_1}\ln\|D^l\|\right\}.$$

Theorem 4.5. *Let the conditions of Theorem 4.1 be satisfied and*

$$e^{-l\widetilde{\beta}^{-}\tau_1} < \|D^l\| < 1.$$

Then a solution to the initial value problem (4.2) satisfies the estimate

$$\|y(t)\| \le \Bigg[\alpha\|D^l\|e^{l\widetilde{\beta}^{-}\tau_1}\left(\|D^l\|e^{l\widetilde{\beta}^{-}\tau_1}-1\right)^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\widetilde{\beta}^{-}\tau_1} + \|D^l\|^{\frac{1}{l}-1}\max\left\{1, \|D\|, \dots, \|D^{l-1}\|\right\}\Phi\Bigg]\exp\left(\frac{t}{l\tau_1}\ln\|D^l\|\right), \qquad t>0,$$

where α, $\widetilde{\beta}^{-}$, and Φ are defined in (2.10) and (4.5).

We prove Theorems 4.3–4.5 in Section 5. Obviously, Theorem 4.1 immediately follows from these theorems.

5 Proof of the main results for (1.1)

As in Section 3, to prove Theorems 4.3–4.5 we need the auxiliary results obtained below.

Lemma 5.1. *Let the conditions of Theorem 4.1 be satisfied. Then, for a solution to the initial value problem (4.2), the following inequality holds*

$$\|y(t)+Dy(t-\tau_1)\| \le \sqrt{\frac{V_0(\varphi)}{h_{\min}(t)}}\exp\left(-\int_0^t \frac{\widetilde{\gamma}(\xi)}{2\|H(\xi)\|}d\xi\right), \quad t>0, \tag{5.1}$$

where $V_0(\varphi)$ and $\widetilde{\gamma}(t)$ are defined by (2.6) and (4.4), respectively, $h_{\min}(t)>0$ is the minimal eigenvalue of the matrix $H(t)$.

Proof. Let $y(t)$ be a solution to the initial value problem (4.2). As above, we consider the Lyapunov–Krasovskii functional (3.2). The time derivative of this functional is

$$\begin{aligned}
\frac{d}{dt}V(t,y) \equiv & \left\langle \frac{d}{dt}H(t)(y(t)+Dy(t-\tau_1)),\ (y(t)+Dy(t-\tau_1)) \right\rangle \\
& + \left\langle H(t)(A(t)y(t)+\sum_{j=1}^{m}B_j(t)y(t-\tau_j)),\ (y(t)+Dy(t-\tau_1)) \right\rangle \\
& + \left\langle H(t)(y(t)+Dy(t-\tau_1)),\ (A(t)y(t)+\sum_{j=1}^{m}B_j(t)y(t-\tau_j)) \right\rangle \\
& + \langle H(t)F(t,y(t),y(t-\tau_1),\ \dots,y(t-\tau_m)),\ (y(t)+Dy(t-\tau_1))\rangle \\
& + \langle H(t)(y(t)+Dy(t-\tau_1)),\ F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m))\rangle \\
& + \sum_{j=1}^{m}\langle K_j(0)y(t),y(t)\rangle - \sum_{j=1}^{m}\langle K_j(\tau_j)y(t-\tau_j),y(t-\tau_j)\rangle \\
& + \sum_{j=1}^{m}\int_{t-\tau_j}^{t}\left\langle \frac{d}{dt}K_j(t-s)y(s),y(s)\right\rangle ds.
\end{aligned}$$

Using the matrix $C(t)$ defined by (2.4), we obtain

$$\begin{aligned}
\frac{d}{dt}V(t,y) \equiv & -\left\langle C(t)\begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t)\end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau_1) \\ z(t)\end{pmatrix}\right\rangle \\
& + \langle H(t)F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m)),\ (y(t)+Dy(t-\tau_1))\rangle \\
& + \langle H(t)(y(t)+Dy(t-\tau_1)),\ F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m))\rangle \\
& + \sum_{j=1}^{m}\int_{t-\tau_j}^{t}\left\langle \frac{d}{dt}K_j(t-s)y(s),y(s)\right\rangle ds,
\end{aligned}$$

where

$$z(t) = \begin{pmatrix} y(t-\tau_2) \\ \vdots \\ y(t-\tau_m)\end{pmatrix}.$$

Taking into account (3.4), we have

$$\begin{aligned}
\frac{d}{dt}V(t,y) \equiv & -\left\langle S(t)\begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t)\end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t)\end{pmatrix}\right\rangle \\
& + \langle H(t)F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m)),\ (y(t)+Dy(t-\tau_1))\rangle \\
& + \langle H(t)(y(t)+Dy(t-\tau_1)),\ F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m))\rangle \\
& + \sum_{j=1}^{m}\int_{t-\tau_j}^{t}\left\langle \frac{d}{dt}K_j(t-s)y(s),y(s)\right\rangle ds,
\end{aligned} \tag{5.2}$$

where the entries of the matrix $S(t)$ are defined in (3.6).

Consider the second and the third summands in the right-hand side of (5.2). In view of (1.2), we obtain

$$\begin{aligned}
J_1(t) &= \langle H(t)F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m)),\ (y(t)+Dy(t-\tau_1))\rangle \\
&\quad + \langle H(t)(y(t)+Dy(t-\tau_1)),\ F(t,y(t),y(t-\tau_1),\dots,y(t-\tau_m))\rangle \\
&\le 2\|H(t)\| \left(q_0\|y(t)\| + \sum_{j=1}^{m} q_j\|y(t-\tau_j)\| \right) \|y(t)+Dy(t-\tau_1)\| \\
&\le 2q_0\|H(t)\|\|y(t)+Dy(t-\tau_1)\|^2 \\
&\quad + 2(q_0\|D\|+q_1)\|H(t)\|\|y(t-\tau_1)\|\|y(t)+Dy(t-\tau_1)\| \\
&\quad + \sum_{j=2}^{m} 2q_j\|H(t)\|\|y(t-\tau_j)\|\|y(t)+Dy(t-\tau_1)\|.
\end{aligned} \tag{5.3}$$

We now show that

$$J_1(t) \le q(t)\left(\|y(t)+Dy(t-\tau_1)\|^2 + \sum_{j=1}^{m} \|y(t-\tau_j)\|^2 \right), \tag{5.4}$$

where $q(t)$ is defined by (4.1). Denote by $J_2(t)$ the right-hand side of (5.3). It can be written out as follows

$$J_2(t) = \|H(t)\| \left\langle V_m \begin{pmatrix} \|y(t)+Dy(t-\tau_1)\| \\ \|y(t-\tau_1)\| \\ \|y(t-\tau_2)\| \\ \vdots \\ \|y(t-\tau_m)\| \end{pmatrix}, \begin{pmatrix} \|y(t)+Dy(t-\tau_1)\| \\ \|y(t-\tau_1)\| \\ \|y(t-\tau_2)\| \\ \vdots \\ \|y(t-\tau_m)\| \end{pmatrix} \right\rangle,$$

where

$$V_m = \begin{pmatrix} 2q_0 & q_0\|D\|+q_1 & q_2 & \cdots & q_m \\ q_0\|D\|+q_1 & 0 & 0 & \dots & 0 \\ q_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_m & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is not hard to verify that

$$\det(V_m - \lambda I) = (-\lambda)^{m-1}\left[\lambda^2 - 2q_0\lambda - (q_0\|D\|+q_1)^2 - q_2^2 - \cdots - q_m^2\right].$$

Consequently, the eigenvalues of V_m have the form

$$\lambda_1 = q_0 + \sqrt{q_0^2 + (q_0\|D\|+q_1)^2 + \sum_{j=2}^{m} q_j^2},$$

$$\lambda_2 = q_0 - \sqrt{q_0^2 + (q_0\|D\|+q_1)^2 + \sum_{j=2}^{m} q_j^2}, \quad \lambda_3 = \cdots = \lambda_{m+1} = 0.$$

Obviously, λ_1 is the maximal eigenvalue. Hence,

$$J_2(t) \le \lambda_1\|H(t)\| \left(\|y(t)+Dy(t-\tau_1)\|^2 + \sum_{j=1}^{m} \|y(t-\tau_j)\|^2 \right),$$

which gives us (5.4).

Consequently,

$$\frac{d}{dt}V(t,y) \le -\left\langle (S(t)-q(t)I)\begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle \\ + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt}K_j(t-s)y(s), y(s) \right\rangle ds. \tag{5.5}$$

By the conditions of Theorem 4.1, the matrix $(S(t)-q(t)I)$ is Hermitian positive definite. Since $S_{23}(t)$ is the zero matrix, it follows from Lemma 3.1 that

$$S(t)-q(t)I = \begin{pmatrix} I & S_{12}(t)(S_{22}(t)-q(t)I)^{-1} & S_{13}(t)(S_{33}(t)-q(t)I)^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ \times \begin{pmatrix} \widetilde{P}(t) & 0 & 0 \\ 0 & S_{22}(t)-q(t)I & 0 \\ 0 & 0 & S_{33}(t)-q(t)I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ (S_{22}(t)-q(t)I)^{-1}S_{12}^*(t) & I & 0 \\ (S_{33}(t)-q(t)I)^{-1}S_{13}^*(t) & 0 & I \end{pmatrix},$$

where

$$\widetilde{P}(t) = S_{11}(t) - q(t)I - S_{12}(t)(S_{22}(t)-q(t)I)^{-1}S_{12}^*(t) - S_{13}(t)(S_{33}(t)-q(t)I)^{-1}S_{13}^*(t);$$

moreover, the matrices

$$\widetilde{P}(t), \quad S_{22}(t)-q(t)I, \quad \text{and} \quad S_{33}(t)-q(t)I$$

are positive definite. Hence,

$$\left\langle (S(t)-q(t)I)\begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix}, \begin{pmatrix} y(t)+Dy(t-\tau_1) \\ y(t-\tau_1) \\ z(t) \end{pmatrix} \right\rangle \\ \ge \left\langle \widetilde{P}(t)(y(t)+Dy(t-\tau_1)), (y(t)+Dy(t-\tau_1)) \right\rangle \ge \widetilde{p}_{\min}(t)\|y(t)+Dy(t-\tau_1)\|^2, \tag{5.6}$$

where $\widetilde{p}_{\min}(t) > 0$ is the minimal eigenvalue of $\widetilde{P}(t)$. Taking into account (3.6), the matrix $\widetilde{P}(t)$ has the form (4.3). Consequently, in view of (5.6), from (5.5) we derive

$$\frac{d}{dt}V(t,y) \le -\widetilde{p}_{\min}(t)\|y(t)+Dy(t-\tau_1)\|^2 + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt}K_j(t-s)y(s), y(s) \right\rangle ds.$$

Using the matrix $H(t)$, we have

$$\|y(t)+Dy(t-\tau_1)\|^2 \ge \frac{1}{\|H(t)\|} \langle H(t)(y(t)+Dy(t-\tau_1)), (y(t)+Dy(t-\tau_1)) \rangle .$$

Hence,

$$\frac{d}{dt}V(t,y) \le -\frac{\widetilde{p}_{\min}(t)}{\|H(t)\|}\langle H(t)(y(t)+Dy(t-\tau_1)),\ (y(t)+Dy(t-\tau_1))\rangle \\ + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} \left\langle \frac{d}{dt}K_j(t-s)y(s), y(s) \right\rangle ds.$$

Repeating the reasoning carried out when proving Lemma 3.2, we obtain (5.1).

The lemma is proved. □

Lemma 5.2. *Let the conditions of Theorem 4.1 be satisfied. Then a solution to the initial value problem* (4.2) *on every segment* $t \in [k\tau_1, (k+1)\tau_1)$, $k = 0, 1, \dots,$ *satisfies the following estimate*

$$\|y(t)\| \leq \alpha \sum_{j=0}^{k} \|D^j\| e^{-\int_0^{t-j\tau_1} \widetilde{\beta}(\xi)\, d\xi} + \|D^{k+1}\|\Phi,$$

where α, $\widetilde{\beta}(t)$, *and* Φ *are defined in* (2.10) *and* (4.5).

Proof. Using Lemma 5.1, this lemma is proved in the same way as Lemma 3.3. □

Proofs of Theorems 4.3–4.5. In the case of one delay, in [11] the analogs of Theorems 4.3–4.5 (see Theorems 2–4 in [11]) were proved in detail by the use of the auxiliary assertions (see Lemmas 2–4 in [11]). In the present paper, using Lemmas 5.1, 5.2 and repeating the reasoning carried out when proving Theorems 2–4 in [11], we derive the required estimates for solutions to the initial value problem (4.2). □

Using the proof of Lemma 5.1, we can reformulate the conditions of exponential stability of the zero solution to the nonlinear system (1.1) as follows.

Theorem 5.3. *Suppose that there exist* $H(t)$, $K_j(s)$, $j = 1, \dots, m$, *satisfying* (2.1)–(2.3) *and such that the matrices*

$$K_1(\tau_1) - D^* \sum_{j=1}^{m} K_j(0) D - q(t) I, \qquad K_j(\tau_j) - q(t) I, \quad j = 2, \dots, m,$$

and $\widetilde{P}(t)$ *defined by* (4.3) *are positive definite for* $t \in [0, T]$. *Then the zero solution to* (1.1) *is exponentially stable.*

Remark 5.4. The results obtained above give us the assertions on robust stability for (1.12). Indeed, consider uncertain systems of the form

$$\frac{d}{dt}\left(y(t) + Dy(t-\tau_1)\right) = (A(t) + \Delta A(t)) y(t) + \sum_{j=1}^{m} (B_j(t) + \Delta B_j(t)) y(t-\tau_j), \tag{5.7}$$

where $\Delta A(t)$, $\Delta B_j(t)$, $j = 1, \dots, m$, are unknown $(n \times n)$ matrices such that

$$\|\Delta A(t)\| \leq q_0, \qquad \|\Delta B_j(t)\| \leq q_j, \quad j = 1, \dots, m.$$

Obviously, in this case the vector function

$$F(t, u, v_1, \dots, v_m) = \Delta A(t) u + \sum_{j=1}^{m} \Delta B_j(t) v_j$$

satisfies (1.2). Then Theorem 4.1 gives us the conditions of robust exponential stability for (1.12). From Theorems 4.3–4.5 we have the estimates of exponential decay of solutions to (5.7).

6 An illustrative example

Consider the system

$$\frac{d}{dt}\left(y(t)+Dy(t-\tau_1)\right)=A(t)y(t)+B_1(t)y(t-\tau_1)+B_2(t)y(t-\tau_2), \tag{6.1}$$

where

$$D=\begin{pmatrix}0.12 & 0.1\\ -0.05 & 0.11\end{pmatrix},\qquad A(t)=\begin{pmatrix}-10+0.2\cos t & 1-0.3\cos t\\ 2+0.5\cos t & -20-0.1\cos t\end{pmatrix},$$

$$B_1(t)=\begin{pmatrix}0.1\sin t & 0.3\cos t\\ 0 & -1\end{pmatrix},\qquad B_2(t)=\begin{pmatrix}0.5\sin 2t & 0\\ -0.4\sin t & 0.2\cos t\end{pmatrix},\quad \tau_1=2,\quad \tau_2=1.$$

According to Theorem 3.4, we can guarantee exponential stability of the zero solution to (6.1) if there exist matrices $H(t)$, $K_1(s)$, $K_2(s)$ satisfying (2.1)–(2.3) and such that

$$K_1(2)-D^*K_1(0)D-D^*K_2(0)D>0$$

and $P(t)$ defined by (2.7) is positive definite for $t\in[0,2\pi]$. We choose the matrices $H(t)$, $K_1(s)$, $K_2(s)$ as follows

$$H(t)=\begin{pmatrix}6-3.4\sin t & 1-2.3\sin t\\ 1-2.3\sin t & 8+2.2\sin t\end{pmatrix},$$

$$K_1(s)=e^{-0.07s}K_0,\quad K_2(s)=e^{-0.28s}K_0,\quad K_0=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}.$$

It is not hard to verify that

$$2.38\le h_{\min}(t)\le 5.65,\qquad 8.4\le\|H(t)\|\le 11.37.$$

Obviously, the matrix

$$K_1(2)-D^*K_1(0)D-D^*K_2(0)D=\begin{pmatrix}0.8305582 & -0.002\\ -0.002 & 1.6703165\end{pmatrix}$$

is positive definite. Calculating $P(t)$, we obtain that

$$3.58\le p_{\min}(t)\le 41.86.$$

Consequently, the zero solution to (6.1) is exponentially stable.

To estimate the decay rate of solutions to (6.1) at infinity we need to calculate $\gamma(t)$ and $\beta(t)$ defined in (2.9) and (2.11), respectively. Obviously, in our case

$$k_1=0.07,\quad k_2=0.28,\quad \gamma(t)=k_1\|H(t)\|.$$

Hence,

$$\beta(t)=\beta^+=\beta^-=k_1/2=0.035.$$

Since $\|D\|<e^{-\beta^+\tau_1}$, by Theorem 2.3 we have the estimate

$$\|y(t)\|\le c\max_{-2\le s\le 0}\|y(s)\|e^{-0.035t},\qquad c>0,\quad t\ge 0.$$

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